

I.  $f(x, y) = \tan^{-1} \frac{y}{x}$  のとき、 $\frac{\partial f}{\partial x}$  及び  $\frac{\partial f}{\partial y}$  を求めよ。

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( \tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( -\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left( \tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

II.  $x = s \cos t$ 、 $y = s \sin t$  とするとき、 $f(x, y)$  について以下の問に答えよ。

1)  $\frac{\partial f}{\partial s}$ 、 $\frac{\partial f}{\partial t}$  を  $\frac{\partial f}{\partial x}$ 、 $\frac{\partial f}{\partial y}$  を用いて表せ。

合成関数の微分法に従って、

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} \cos t + \frac{\partial f}{\partial y} \sin t$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} (-s \sin t) + \frac{\partial f}{\partial y} s \cos t = -\frac{\partial f}{\partial x} s \sin t + \frac{\partial f}{\partial y} s \cos t$$

2)  $\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 = \left( \frac{\partial f}{\partial s} \right)^2 + \frac{1}{s^2} \left( \frac{\partial f}{\partial t} \right)^2$  が成り立つことを示せ

$$\left( \frac{\partial f}{\partial s} \right)^2 + \frac{1}{s^2} \left( \frac{\partial f}{\partial t} \right)^2 = \left( \frac{\partial f}{\partial x} \cos t + \frac{\partial f}{\partial y} \sin t \right)^2 + \frac{1}{s^2} \left( -\frac{\partial f}{\partial x} s \sin t + \frac{\partial f}{\partial y} s \cos t \right)^2$$

$$= \left( \frac{\partial f}{\partial x} \cos t \right)^2 + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cos t \sin t + \left( \frac{\partial f}{\partial y} \sin t \right)^2 + \frac{1}{s^2} \left( \left( \frac{\partial f}{\partial x} s \sin t \right)^2 - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} s^2 \cos t \sin t + \left( \frac{\partial f}{\partial y} s \cos t \right)^2 \right)$$

$$= \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2$$